Properties of the Langevin and Fokker-Planck equations for scalar fields and their application to the dynamics of second order phase transitions

Luís M. A. Bettencourt
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
(Received 31 May 2000; published 29 January 2001)

I consider several Langevin and Fokker-Planck classes of dynamics for scalar field theories in contact with a thermal bath at temperature *T*. These models have been applied recently in the numerical description of the dynamics of second order phase transitions and associated topological defect formation as well as in other studies of the dynamics of critical phenomena. Closed form solutions of the Fokker-Planck equation are given for a harmonic potential and a dynamical mean-field approximation is developed. These methods allow for an analytical discussion of the behavior of the theories in several circumstances of interest such as critical slowing down at a second order transition and the development of spinodal instabilities.

DOI: 10.1103/PhysRevD.63.045020 PACS number(s): 11.30.Qc, 02.50.−r, 05.70.Ce, 98.80.Cq

I. INTRODUCTION

The dynamics of systems undergoing a second order symmetry breaking phase transition has been studied recently in several instances [1–5], particularly in association with the theory of topological defect formation [6,7] in cosmology [8] and in experiments in ^{3}He [9] and ^{4}He [10]. With experimental data in the pipeline for the collisions of heavy ions at the BNL Relativistic Heavy Ion Collider RHIC and later at CERN the effort to identify phase transitions in nuclear matter is also a very active field of research. The Langevin dynamics of effective scalar theories (such as the σ model) affords one of the few quantitative windows into such transitions [11].

Theoretically this situation has been modeled by the out-of-equilibrium dynamics of a classical scalar field theories (with a given number of flavors) in contact with an external environment at a temperature *T*, which may be a function of time. The environment is only known statistically and its behavior can then be described in terms of stochastic fields obeying a fluctuation-dissipation relation. The effective evolution of the fields is therefore described by a Langevin field equation.

This stochastic classical description is of course only an approximation to the full quantum evolution [12]. Recently attempts to treat quantum field theories dynamically [13] have been made but the possibilities afforded by these approximations thus far do not allow for the correct description of thermalization processes [14] (with the exception of Boltzmann transport theory [15]) and have therefore very limited applicability to the description of the dynamics of a system undergoing a symmetry breaking phase transition.

In addition a reasonable case can be made that close to a second order transition the theory is effectively at high temperature $T \gg m(T)$, where $m(T)$ is the temperature dependent mass scale, and its infrared should behave approximately classically. In this regime some quantum effects can also be included effectively through the appropriate choice of the coupling of the fields to the external environment (the quantum fluctuations in this case) and/or their contributions to the mass and couplings of the dynamical fields. This line of considerations [16] has been used recently to describe the effective evolution of the long wave-length modes of the color fields of non-Abelian gauge theories in the high temperature regime, as a stochastic classical field theory.

In cosmology and high energy physics field theories are relativistic. This implies that the time evolution of scalar fields is governed by a second order derivative in time. The canonical Langevin equations however are usually formulated in terms of a first derivative evolution in time. This difference is unimportant in equilibrium, but modifies real-time properties. In certain circumstances, which I discuss in detail below, the second order evolution effectively leads to a first order (or overdamped) description of certain parts of the system.

The general properties of Langevin field evolutions are that, under very minimal conditions, the fields equilibrate for long times to the canonical (classical) Boltzmann distribution. The dynamics are very rich, realizing and generalizing all the near canonical equilibrium properties of the classical field evolution, including in particular the features captured by the perhaps more familiar time-dependent Ginzburg Landau formulation (TDGL).

In this paper I present a didactic introduction to the underlying formulation of a second time derivative Langevin system and its associated Fokker-Planck equation. Its general dynamical properties are compared to those of other important classes of dynamics such as TDGL and the stochastic non-linear Schrödinger equation (SNLS). This is done mostly in Sec. II. (The treatment of the SNLS system is given in Appendix A.) In Sec. III I derive analytic solutions of the Fokker-Planck equation in the particular case of a harmonic potential. These solutions are fairly simple but to the best of my knowledge have not been discussed in the literature of stochastic field theories. A mean-field formulation of the dynamical problem is given in Sec. IV and it is found that its solution is similar in effort to solving the associated Langevin mean-field equation, but with the stochastic averages already taken into account. In Sec. V I discuss applications of the solutions of Secs. III and IV to the understanding of the full non-linear dynamics of the system. Although qualitative it will be argued that these solutions capture the essentials of the full dynamics in the critical domain and permit an understanding of the relevant thermalization times. A brief discussion of spinodal instabilities is also included. Finally in Sec. VI I summarize the conclusions of this paper.

II. THE LANGEVIN AND FOKKER-PLANCK EQUATIONS FOR SCALAR FIELDS

We start by formulating the problem. The second order Langevin equation for a scalar fields $\phi(x,t)$ with an arbitrary interaction potential $V[\phi]$ is given by

$$(\partial_t^2-\nabla^2)\phi(x)+\frac{\delta V(\phi)}{\delta\phi(x)}+\eta\dot{\phi}(x)=\xi(\mathbf{x},t). \tag{1}$$

The stochastic fields $\xi(x,t)$ are taken to be Gaussian and white, characterized by

$$\langle\xi(\mathbf{x},t)\rangle=0, \quad \langle\xi(\mathbf{x},t)\xi(\mathbf{x}',t')\rangle=\Omega\,\delta^3(\mathbf{x}-\mathbf{x}')\,\delta(t-t'). \tag{2}$$

Although we have written this equation for a single scalar field $\phi(x,t)$ the generalization of Eqs. (1)–(2) and of what follows to a $O(N)$ symmetric theory, with N the number of flavors, is straightforward.

We can proceed to reduce the order of this differential system, by introducing field generalized momenta conjugate to ϕ:

$$\partial_t\phi(x)=\pi(x), \tag{3}$$

$$\partial_t\pi(x)=-\eta\pi(x)+\nabla^2\phi(x)-\frac{\delta V(\phi)}{\delta\phi(x)}+\xi(x). \tag{4}$$

An interesting limit of Eqs. (3)–(4) arises when η is large, or more precisely when $\partial_t\pi(x)\ll-\eta\pi(x)$. Then we can write

$$\eta\partial_t\phi(x)=\nabla^2\phi(x)-\frac{\delta V(\phi)}{\delta\phi(x)}+\xi(x). \tag{5}$$

Although this is the more conventional form of the Langevin field equation [17,18], I will refer to it as the overdamped limit of Eqs. (3)–(4).

The objective of the Langevin analysis is to measure the expectation value (over the stochastic fields) of any functional ρ of the fields $\pi(x),\phi(x)$, generated by the evolution of Eqs. (1-2). With the choice Eq. (2) this average is a Gaussian functional integral of the form

$$\int D\xi\rho[\pi_\xi,\phi_\xi]e^{-(1/2\Omega)\int d^4x\xi^2} \tag{6}$$

where the ξ subscripts in the field and its momentum denote their functional dependence on the stochastic fields $\xi(x,t)$, through the evolution of Eqs. (3)–(4). Relation (6) and (2) can be generalized to fields ξ with more complicated Gaussian distributions.

In the presence of the stochastic fields $\xi(x,t)$ the initial conditions for the fields $\pi(x)$, $\phi(x)$ may only be know statistically themselves. They can then be expressed in terms of a functional probability distribution $P[\pi,\phi]$, at the initial time.

Throughout the evolution we assume that a time dependent probability distribution $P_{\rm FP}[\phi(x),\pi(x),t]$, a functional of the time independent fields and a function of time, exists. This is the Fokker-Planck probability distribution, which we require, as usual, to be positive definite and normalizable in the sense

$$\mathcal{N}\int D\pi D\phi P_{FP}[\phi,\pi,t]=1, \tag{7}$$

where $\mathcal{N}$ is independent of the fields π, ϕ but may be time dependent. Note that in general $\mathcal{N}$ as well as other expectation values of the fields may be formally infinite as the ultraviolet cutoff of the theory is taken to zero.

The time dependent expectation value over the stochastic fields ξ of any quantity $\rho[\phi,\pi]$ is then given by

$$\langle\rho\rangle(t)=\mathcal{N}\int D\pi D\phi P_{FP}[\phi,\pi,t]\rho[\phi,\pi]. \tag{8}$$

In order to be useful this picture requires the explicit knowledge of $P_{FP}[\phi,\pi,t]$. The Langevin equation generates a time-dependent (Fokker-Planck) probability distribution $P_{FP}[\pi,\phi,t]$, which we can write formally

$$P_{FP}[\pi,\phi,t]=\left\langle\int d^Dx\,\delta[\hat{\pi}(x,t)-\pi(x)]\times\delta[\hat{\phi}(x,t)-\phi(x)]\right\rangle, \tag{9}$$

where the brackets denote, as before, average over the stochastic fields. The static fields are the arguments of $P_{FP}[\pi,\phi,t]$, whereas $\hat{\pi}(x,t)$, $\hat{\phi}(x,t)$ obey the Langevin Eqs. (3),(4).

The probability $P_{FP}[\pi,\phi,t]$, obeys a functional Fokker-Planck evolution equation that can be computed directly by differentiating Eq. (9). Following Zinn-Justin [17]

$$\partial_t P_{FP}=\left\langle\left[\int d^Dx\partial_t\hat{\phi}(x,t)\frac{\delta}{\delta\hat{\phi}(x,t)}+\partial_t\hat{\pi}(x,t)\frac{\delta}{\delta\hat{\pi}(x,t)}\right]\times\delta[\hat{\pi}(x,t)-\pi(x)]\delta[\hat{\phi}(x,t)-\phi(x)]\right\rangle. \tag{10}$$

The properties of the δ allow us to trade the functional derivatives of $\hat{\pi}(x,t),\hat{\phi}(x,t)$ for derivatives relative to $\pi(x),\phi(x)$. These can be taken out of the stochastic average $\langle\ \dots\ \rangle$. The only remaining term inside this average results from the appearance of the stochastic fields ξ in the equation of motion for $\hat{\pi}(x,t)$. This term is of the form

$$\left\langle \int d^D x \xi(x,t) \delta[\hat{\pi}(x,t)-\pi(x)] \right\rangle. \tag{11}$$

The Gaussianity of the stochastic fields enforces the identity [17]

$$\langle \xi(x,t) \delta[\hat{\pi}(x,t)-\pi(x)] \rangle$$
$$= \Omega \left\langle \frac{\delta}{\delta \xi(x,t)} \delta[\hat{\pi}(x,t)-\pi(x)] \right\rangle$$
$$= \Omega \left\langle \frac{\delta \hat{\pi}(x,t)}{\delta \xi(x,t)} \frac{\delta}{\hat{\pi}(x,t)} \delta[\hat{\pi}(x,t)-\pi(x)] \right\rangle. \tag{12}$$

Once again we can trade the functional derivative relative to $\hat{\pi}(x,t)$ for another relative to $\pi(x)$. The expectation value of $\delta\hat{\pi}(x,t)/\delta\xi(x,t)$ can in turn be obtained from the formal integration of the equation of motion by using a regularized delta function in time. We must effectively take half of this delta function to obtain

$$\frac{\delta \hat{\pi}(x,t)}{\delta \xi(x,t)} = \frac{1}{2}. \tag{13}$$

Bringing together all the terms results finally in

$$\partial_t P_{\mathrm{FP}}[\pi,\phi,t] = -\mathcal{H}_{\mathrm{FP}} P_{\mathrm{FP}}[\pi,\phi,t], \tag{14}$$

where

$$\mathcal{H}_{\mathrm{FP}} = -\frac{\Omega}{2}\frac{\delta^2}{\delta\pi^2} + \pi\frac{\delta}{\delta\phi} - \eta - \left(\eta\pi - \nabla^2\phi + \frac{\delta V(\phi)}{\delta\phi}\right)\frac{\delta}{\delta\pi}. \tag{15}$$

It is useful to think in terms of the functional operator $\mathcal{H}_{\mathrm{FP}}$ as the generator of infinitesimal time displacements of the probability functional P_{FP}. If, as in most applications, the potential $V(\phi)$ is explicitly time independent we can invoke a separation ansatz for P_{FP} such that

$$P_{\mathrm{FP}}[\pi,\phi,t] = \mathcal{P}[\pi,\phi] T(t). \tag{16}$$

Thus we can regard Eq. (14) as an analog of a functional Schrödinger equation, in imaginary time. Then we can write the time independent and dependent equations

$$\mathcal{H}_{\mathrm{FP}} \mathcal{P}_n = E_n \mathcal{P}_n, \quad \partial_t T(t) = -E_n T(t). \tag{17}$$

The functional dependence on the fields is now limited to the static probability eigenfunctionals $\mathcal{P}_n$. The time evolution of the Fokker-Planck distribution is completely characterized by the spectrum of eigenvalues of $\mathcal{H}_{\mathrm{FP}}$, E_n. An orthogonal complete basis of functionals B_n can in general be constructed from the set $\mathcal{P}_n$.[1] Note that it is not possible in general to separate Eq. (16) further into two equations, one in π and another in ϕ via an ansatz like $P_{\mathrm{FP}} = R[\pi] S[\phi]$.

On general grounds we expect a steady state corresponding to thermal equilibrium to be reached for long times. Formally, we can then project the evolution of P_{FP} in terms of the eigenvalues E_n and functionals B_n as

$$P_{\mathrm{FP}}[\pi,\phi,t] = \sum_{n=0}^{\infty} C_n B_n[\pi,\phi] e^{-E_n t}. \tag{18}$$

where the C_i's are the projections of P_{FP} at the initial time onto the basis of eigenfunctionals B_n:

$$C_n = \int D\pi D\phi B_n[\pi,\phi] P_{\mathrm{FP}}[\pi,\phi,t=0], \tag{19}$$

where both B_n and $P_{\mathrm{FP}}[\pi,\phi,t]$ are taken to be normalized.

If equilibrium is approached for long times the corresponding distribution must be associated with the zero mode of $\mathcal{H}_{\mathrm{FP}}$, $E_0 = 0$, i.e., it is stable. Contributions from eigenstates $\mathcal{P}_n$, $n \neq 0$, vanish exponentially, in a characteristic time $t_{\mathrm{eq}} \sim E_n^{-1}$, provided that the real part of E_n is positive. This follows from the requirement that $V(\phi)$ is bounded from below. For the E_n, solutions of the second time derivative eigenproblem there may be in addition an imaginary part. Thus the excited states decay away in time as an exponentially damped oscillator.

The eigenfunctional $\mathcal{P}_0$, characterizing the long time field probability is then interpreted as the equilibrium distribution. Taking $E_0 = 0$ in Eq. (17) we find

$$\mathcal{P}_0[\pi,\phi] = \mathcal{N} \exp\left[-\beta \int d^D x \left(\frac{\pi(x)^2}{2} + S(\phi)\right)\right] \tag{20}$$

with $S(\phi) = (\frac{1}{2}[\nabla\phi(x)]^2 + V(\phi))$. Here we took $\Omega = 2\eta/\beta$, which is the analog of Einstein's relation for Brownian motion, and ensures the balance between fluctuations [sourced by the fields $\xi(x,t)$] and dissipation [through the $-\eta\pi(x,t)$ in Eq. (4)] at long times. This solution is not unique in general. To determine the remaining ambiguities we must invoke boundary conditions and normalization. The condition on the probability distribution to be normalizable (i.e., to vanish sufficiently fast in the limit of large fields) is usually sufficient to eliminate solutions different from Eq. (20) [19].

[1]The set of functionals $\mathcal{P}_n$ will only be guaranteed to be an orthogonal basis if $\mathcal{H}_{\mathrm{FP}}$ is an Hermitian functional operator, which is not true in general. The non-Hermiticity of $\mathcal{H}_{\mathrm{FP}}$ is necessary on physical grounds since the Langevin evolution deals with a system exchanging energy with a ''reservoir,'' which leads ultimately to its thermalization in the canonical ensemble (vs the microcanonical ensemble, if energy were conserved). The relation between the B_n and P_n is usually very simple involving a power of the canonical distribution and it follows that E_n are also the eigenvalues of the set B_n. See, e.g., [19] for some examples in few degree of freedom problems.

It is interesting to note that it can be read directly from Eq. (20) that the momentum equilibrium variance

$$\langle \pi(x)^2\rangle = 1/\beta = T, \tag{21}$$

which expresses equipartition in a relativistic classical field theory. This quantity is also an excellent thermometer for the dynamical evolution.

Now, because the canonical distribution is Gaussian in π we can proceed to obtain a reduced distribution in terms of ϕ alone. Performing the Gaussian integral over all π we obtain

$$P_{eq}[\phi] = \mathcal{N}' \exp\left[-\beta \int d^D x S(\phi)\right], \tag{22}$$

which is the canonical Boltzmann distribution. $P_{eq}[\phi]$ could have been equally obtained from the static solution of the Fokker-Planck equation associated with the perhaps more usual first derivative Langevin equation (see below). In this respect we see that *in equilibrium* it makes no difference to perform the evolution, with or without the second time derivative in Eq. (1). Away from equilibrium, or even if we simply wish to study the fluctuations around it, we should take the appropriate form of the evolution, choosing to keep or neglect the second time derivative according to the physical picture under consideration. The difference between both of these evolutions, is expressed in terms of the higher eigenvalues and eigenfunctionals associated with the different Hamiltonians, which I discuss below.

For a Gaussian potential $V[\phi] = (m^2/2)\phi^2$ we can also read the equilibrium thermal propagator directly from Eq. (22). It is, in momentum space

$$\langle \phi_p \phi_{-p}\rangle = \frac{T}{p^2+m^2}. \tag{23}$$

This is the canonical free Boltzmann propagator, which is the correct description for ideal classical fields at finite temperature.

It is illuminating to compare the above framework directly with that for the overdamped Langevin evolution. Following the same, but somewhat simpler, procedure it can be shown [18] that the Fokker-Planck Hamiltonian takes the form

$$\mathcal{H}_{\rm FP} = -\frac{\delta}{\delta\phi}\left[\frac{\Omega}{2\eta}\frac{\delta}{\delta\phi} - \nabla^2\phi + \frac{\delta V}{\delta\phi}\right], \tag{24}$$

leading to the eigenvalue functional equation

$$-\left\{\frac{\delta}{\delta\phi}\left[\frac{\Omega}{2\eta}\frac{\delta}{\delta\phi} - \nabla^2\phi + \frac{\delta V}{\delta\phi}\right] + E_N\right\} P[\phi] = 0, \tag{25}$$

$$\eta\partial_t P[\phi] = -E_N P[\phi]. \tag{26}$$

Again the canonical distribution is the eigenfunctional P_0 corresponding to a zero eigenvalue, i.e.,

$$E_0 = 0; \quad P_{eq}[\phi] = \mathcal{N}' e^{-\beta\int d^D x (1/2)(\nabla\phi)^2 + V(\phi)}, \tag{27}$$

where as before the Einstein relation $\Omega = 2\eta/\beta$ must hold.

III. APPROACH TO EQUILIBRIUM

A complete dynamical solution of the ensemble (under average over the noise) of fields obeying the Langevin equation (3),(4) can be obtained in principle by solving the corresponding Fokker-Planck equation. Unfortunately the task of solving for these non-linear functional equations in general is quite monumental. The main difficulty is connected with the non-local form of the equations: in x-space this arises from the Laplacian term, while in k space the difficulty is transferred to the non-linear terms.

Below I give closed form solutions to the equations in the fields and their conjugate momenta for the particular case of the harmonic potential. These are still interesting since they offer an alternative to the direct solutions of the Langevin equation, which, clearly, can also be found in the linear case. In the former picture, however, the averages over the stochastic fields are already taken into account.

It is simpler to build some intuition for the solution of the overdamped Fokker-Planck equation first to which I now turn.

A. The overdamped Fokker-Plank equation and its solutions

The advantage of the harmonic case is that in Fourier space different modes decouple in the Langevin equation:

$$\eta\partial_t\phi_k(t) = -(k^2+m^2)\phi_k(t) + \xi_k(t) \tag{28}$$

where the stochastic fields obey

$$\langle \xi_k(t)\rangle = 0, \quad \langle \xi_k(t)\xi_{-k}(t')\rangle = \Omega\,\delta(t-t') \tag{29}$$

and

$$\phi(x,t) = \int \frac{d^D k}{(2\pi)^D}\phi_k(t) e^{ik\cdot x}. \tag{30}$$

The fact that $\phi(x,t)$ is real implies $\phi_k = \phi^*_{-k}$ and similarly for ξ_k. Because of these relations it is more convenient to work with $\phi_k^R = \mathrm{Re}(\phi_k)$ and $\phi_k^I = \mathrm{Im}(\phi_k)$, which obey the Langevin Eq. (28) with $\xi_k^{R,I}$:

$$\langle \xi_k(t)^{R,I}\rangle = 0, \quad \langle \xi_k^{R,I}(t)\xi_k^{R,I}(t')\rangle = \frac{\Omega}{2}\delta(t-t'). \tag{31}$$

The Fokker-Planck equation for the modes now is

$$\eta\partial_t P_k = -\mathcal{H}_k P_k \tag{32}$$

$$\mathcal{H}_k = -\frac{\delta}{\delta\phi_k^{R,I}}\left[\frac{\Omega}{4\eta}\frac{\delta}{\delta\phi_k^{R,I}} + (k^2+m^2)\phi_k^{R,I}\right] \tag{33}$$

which holds, as indicated, both for the real and imaginary components of ϕ_k. The solution of the Fokker-Planck equation then follows as

$$P = \Pi_{k=0}^{\infty}[P_k(\phi_k^R)P_k(\phi_k^I)]. \tag{34}$$

The time independent solution is clearly

$$P_k(\phi_k^R)P_k(\phi_k^I)=\exp[-\beta(k^2+m^2)(\phi_k^{R2}+\phi_k^{I2})] \quad (35)$$

which leads to

$$\begin{aligned}P_0&=\exp\left[-\beta\int_0^\infty \frac{d^Dk}{(2\pi)^D}(k^2+m^2)(\phi_k^{R2}+\phi_k^{I2})\right]\\&=\exp\left[-\beta\int_{-\infty}^{+\infty}\frac{d^Dk}{(2\pi)^D}\phi_k\frac{k^2+m^2}{2}\phi_{-k}\right]\\&=\exp\left[-\beta\int_{-\infty}^{\infty}d^Dx\,\frac{1}{2}(\nabla\phi(x))^2+\frac{m^2}{2}\phi^2(x)\right]. \quad (36)\end{aligned}$$

To compute the eigenfunctions corresponding to non-zero eigenvalues we need to solve the eigenvalue problem

$$\left[\frac{1}{2\beta}\frac{\delta^2}{\delta\phi_k^{R,I2}}-(k^2+m^2)\phi_k^{R,I}\frac{\delta}{\delta\phi_k^{R,I}}+E_N\right]F_N=0 \quad (37)$$

where we wrote $P_k=P_0F_N$. This equation has a familiar solution. To see this explicitly we perform a change of variables to obtain

$$\left[\frac{\delta^2}{\delta X}-2X\frac{\delta}{\delta X}+\frac{2E_N}{k^2+m^2}\right]F_N=0 \quad (38)$$

where $X=\sqrt{\beta(k^2+m^2)}\phi_k^{R,I}$. Equation (38) has the solution

$$E_N=N(k^2+m^2) \quad (39)$$

$$F_N=H_N(X)=H_N(\sqrt{\beta(k^2+m^2)}\phi_k^{R,I}) \quad (40)$$

where H_N is the Hermite polynomial of order N.

From the properties of the Hermite polynomials we know that these solutions are orthogonal under the measure e^{X^2}, which is just the k-part of the canonical distribution. The set of functions $\{H_N(X)e^{X^2/2}\}$ thus constitutes an orthogonal basis. It can additionally be shown trivially that this basis is complete.

The non-zero eigenvalues E_N set the time scales for the approach to thermal equilibrium, which we will discuss in more detail below.

B. The second order Fokker-Planck equation

For the second time-derivative harmonic evolution the Langevin equation becomes

$$\partial\phi_k=\pi_k \quad (41)$$

$$\partial\pi_k=-\eta\pi_k-(k^2+m^2)\phi_k+\xi_k, \quad (42)$$

which as in Sec. III A can be written in terms of real and imaginary components. The Fokker-Planck Hamiltonian for the latter is

$$\begin{aligned}\mathcal{H}_k=&-\frac{\Omega}{4}\frac{\delta^2}{\delta\pi_k^{R,I^2}}+\pi_k^{R,I}\frac{\delta}{\delta\phi_k^{R,I}}\\&-\frac{\delta}{\delta\pi_k^{R,I}}[\eta\pi_k^{R,I}+(k^2+m^2)\phi_k^{R,I}]. \quad (43)\end{aligned}$$

To find the excited states we proceed as before to factor out the canonical distribution $P=F_NP_0$ to get

$$\left\{-\frac{\Omega}{4}\frac{\delta^2}{\delta\pi_k^{R,I^2}}+[\eta\pi_k^{R,I}-(k^2+m^2)\phi_k^{R,I}]\frac{\delta}{\delta\phi_k^{R,I}}+\frac{\delta}{\delta\phi_k^{R,I}}-E_N\right\}F_N=0. \quad (44)$$

We can bring Eq. (44) to the form of Eq. (38) by making the change of variables $X_\pm=a_\pm\pi_k^{R,I}+b_\pm\phi_k^{R,I}$, with

$$a_\pm^2=\frac{\beta}{2}\left[1\pm\sqrt{1-4\frac{k^2+m^2}{\eta^2}}\right], \quad (45)$$

$$b_\pm^2=\frac{2\beta}{\eta^2}(k^2+m^2)^2\Bigg/\left[1\pm\sqrt{1-4\frac{k^2+m^2}{\eta^2}}\right]. \quad (46)$$

The solutions of Eq. (44) then are

$$\begin{aligned}E_N^\pm&=N\frac{a_\pm^2\eta}{\beta}=N\frac{\eta}{2}\left[1\pm\sqrt{1-4\frac{k^2+m^2}{\eta^2}}\right]\\F_N^\pm&=H_N[X_\pm].\end{aligned} \quad (47)$$

These solutions can be brought to products of Hermite polynomials in $\pi_k^{R,I}$ and $\phi_k^{R,I}$ alone through the use of the property

$$H_N[x+y]=2^{-n/2}\sum_{i=0}^{N}\binom{i}{N}H_{N-i}[x\sqrt{2}]H_i[y\sqrt{2}]. \quad (48)$$

This property allows us to see that the solutions

$$P_N=(F_N[X_+]e^{-E_N^+t}+F_N[X_-]e^{-E_N^-t})P_0 \quad (49)$$

form a complete orthogonal basis.

A few properties of these solutions are worth pointing out. The coefficients $a_\pm, b_\pm$ and the eigenvalues $E_N^\pm$ can now be complex numbers. This happens for $4(k^2+m^2)>\eta^2$. In this case $X_+=X_-^*$, $E_N^+=(E_N^-)^*$ and the combination in Eq. (49) remains real, as required of a probability distribution function.

The overdamped limit is recovered as $\eta^2\gg k^2+m^2$. Then

$$E_N^-=N\frac{k^2+m^2}{\eta}, \quad (50)$$

$$a_-^2=\frac{\beta}{2}\frac{k^2+m^2}{\eta^2}\ll b_-^2=\frac{\beta}{2}(k^2+m^2). \quad (51)$$

The eigenfunctionals associated with $E_N^+ \simeq N\eta[1+(k^2+m^2)/\eta^2]$ are, in contrast, damped away rapidly.

The characteristic thermalization times of the system are thus set by $t_N = 1/E_N$. For the long wave length modes ($k \simeq 0$) we have that the equilibration time $t_{eq} \sim \eta/(k^2+m^2)$. In the converse limit, of short wavelengths, the decay of all states is controlled at leading order by $t_{eq} \sim 2/\eta$, which is scale independent. In this sense the evolution of the short wavelength modes is quantitatively different from the overdamped Langevin system.

IV. A MEAN-FIELD APPROXIMATION TO THE NON-LINEAR FOKKER-PLANCK DYNAMICS

It would no doubt be desirable to be able to solve the Fokker-Planck equations for more general (interacting) potentials. At the next level of complexity it is possible to attempt a solution of the Fokker-Planck equation in a mean-field approximation, which we discuss in this section.

In order to illustrate the procedure let us consider an interaction potential of the form $V_{int} = (\lambda/4)\phi(x)^4$. In momentum space this potential generates a term in the Fokker-Planck equation for ϕ_k of the form

$$\frac{\delta V_{int}}{\delta \phi_k} = \lambda \int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D}\ \phi_p \phi_q \phi_{k-p-q}. \tag{52}$$

A mean-field approximation to this term can be obtained by taking $p = -q$ and the noise average such that

$$\frac{\delta V_{MF}}{\delta \phi_k} = 3\lambda \left[\int \frac{d^D p}{(2\pi)^D} \langle \phi_p \phi_{-p} \rangle \right] \phi_k \equiv 3\lambda G(t) \phi_k. \tag{53}$$

Under this approximation the evolution equation can be written in terms of a mean-field Fokker-Planck operator where V_{MF} substitutes the potential V, with

$$V_{MF} = V_0 + \frac{3\lambda}{4} G(t)^2, \tag{54}$$

where V_0 corresponds to the purely harmonic case, together with the self-consistency condition

$$G(t) = \int \frac{d^D p}{(2\pi)^D} \int d\phi_p\ \phi_p \phi_{-p} P[\pi_q, \phi_p, t]. \tag{55}$$

It is then clear that under this Gaussian approximation the theory remains harmonic, but with a time-dependent (self-consistently determined) mass.

Unfortunately the time dependence of the mean field potential destroys the separability of the solution into a function of time, and a functional of the fields and their conjugate momenta. Closed form solutions are thus difficult to construct, which is of course also the case for the Langevin equation under a similar approximation. Numerical solutions of the this mean field Fokker Planck equation can nevertheless be easily obtained.

As a starting point let us again consider the overdamped case. Mean field approximations correspond in general to Gaussian probability distributions. In the overdamped case, this can only lead to

$$P_{MF}[\phi_k, t] = \mathcal{N}(t) \exp\left[-\phi_{-k} \frac{A_k^2(t)}{2} \phi_k \right], \tag{56}$$

where the normalization $\mathcal{N}(t)$ is a function of $A_k^2(t)$

$$\mathcal{N}_k(t) = A_k(t)/\sqrt{\pi}, \tag{57}$$

and must therefore be time-dependent. This does not of course introduce any additional dynamical freedom.

The mean-field Fokker-Planck equation for $P_{MF}[\phi_k, t]$ translates into an equation for the coefficients $A_k^2(t)$ of the form

$$\eta \frac{dA_k}{dt} = [k^2 + m^2 + 3\lambda G(t) - A_k^2 T] A_k, \tag{58}$$

$$G(t) = \int \frac{d^D p}{(2\pi)^D} \frac{1}{A_p^2}. \tag{59}$$

Clearly solving the Fokker-Planck equation in this mean-field approximation requires the solution of a system of coupled ordinary differential equations, one for every mode k. This is similar in effort to the solution of the Langevin mean-field equation with the advantage that the stochastic average is already taken into account. This latter step corresponds to the functional integral necessary to compute $G(t)$ in the Fokker-Planck picture, which can be given in closed form because P_{MF} is Gaussian.

The canonical (mean-field) distribution is of course a static solution of Eqs (58-59) where A_k obeys

$$A_k^2 = \frac{k^2+m^2}{T} + 3\frac{\lambda}{T} \int \frac{d^D p}{(2\pi)^D} \frac{1}{A_p^2}. \tag{60}$$

It is then clear that for vanishing interactions $\lambda = 0$, the free Boltzmann propagator is the solution of Eq. (60). In general because the mean-field contribution is momentum independent Eq. (60) is equivalent to the ''gap'' equation

$$A_k^2 = \frac{k^2+m^2+\Delta m^2}{T} \tag{61}$$

$$\Delta m^2 = 3\lambda \int \frac{d^D p}{(2\pi)^D} \frac{T}{k^2+m^2+\Delta m^2}. \tag{62}$$

For the second order Fokker-Planck equation the most general Gaussian function of the field and momentum modes is of the form

$$P_{MF}[\pi_k,\phi_k,t]=\mathcal{N}_k(t)\exp\left\{-\left[\frac{A_k^2}{2}\phi_k\phi_{-k}+\frac{B_k^2}{2}\pi_k\pi_{-k}+C_k^2\,\mathrm{Re}(\pi_k\phi_{-k})\right]\right\}. \tag{63}$$

The normalization is now a function of all A_k^2, B_k^2, C_k^2

$$\mathcal{N}_k=\frac{1}{\pi}\sqrt{A_k^2B_k^2-C_k^4}. \tag{64}$$

The equations of motion for A_k^2, B_k^2 and C_k^2 then follow from the second order Fokker-Planck equation:

$$\frac{dA_k^2}{dt}=2C_k^2[k^2+m^2+3\lambda G(t)-\eta T C_k^2]$$

$$\frac{dB_k^2}{dt}=2\eta B_k^2[1-TB_k^2]-2C_k^2 \tag{65}$$

$$\frac{dC_k^2}{dt}=\eta C_k^2[1-2TB_k^2]-[A_k^2-(k^2+m^2+3\lambda G(t))B_k^2],$$

where

$$G(t)=\int\frac{d^Dk}{(2\pi)^D}\frac{B_k^2}{A_k^2B_k^2-C_k^4}. \tag{66}$$

This system has as static solutions the mean-field canonical distribution namely

$$A_k^2=\frac{k^2+m^2+\Delta m^2(T)}{T}, \tag{67}$$

$$B_k^2=1/T=\beta \tag{68}$$

$$C_k^2=0, \tag{69}$$

where the equation for A_k is simply Eq. (60). Other static solutions exist though. Additional stationary distributions distinct from thermal equilibrium also occur in the microcanonical evolution of the class of models considered here [14] and seemingly result from the truncation of the system at any finite order in a hierarchy of correlators. The overdamped limit, $k^2+m^2+\Delta m^2(t)\ll\eta$ is obtained when $C_k^2\simeq A_k^2/\eta \ll A_k^2$, $B_k^2\ll A_k^2$.

Again we see that the effort in solving the mean-field second order Fokker-Plank equation is comparable to its Langevin counterpart, but with the stochastic averages taken into account. A renormalization scheme for rendering these equations ultraviolet cutoff independent can be constructed, if desired, following standard techniques for mean-field evolutions [13].

V. APPLICATIONS OF FOKKER-PLANCK SOLUTIONS

The solutions presented above permit some analysis of several important dynamical situations.

In many cases in field theory the harmonic solutions can reveal the essential physics of the (short time) evolution. This is true for example for the motion of fields in the vicinity of a second order phase transition, where only the lowest relevant operators are needed, and for some situations when (spinodal) instabilities can develop, such as in the early stages of reheating after cosmological inflation or in the aftermath of a violent pressure quench, in some condensed matter experiments. Several recent numerical studies have addressed these questions using stochastic scalar field theories, see e.g. [1–4,20–22]. The discussion below allows some insights into their findings.

In this Sec. I analyze these two situations in the context of the solutions of Secs. III and IV.

A. Critical dynamics of the fields and the theory of topological defect formation at second order transitions

In the vicinity of a symmetry breaking second order transition the physical mass squared of the fields will approach zero, with a universal critical exponent ν,

$$m^2(T)\sim m^2\left|\frac{T-T_c}{T_c}\right|^{\nu},$$

where T_c is the critical temperature. This can be seen already at first order in perturbation theory, albeit only with the mean-field value of $\nu=1/2$. In 3D we have

$$m^2(T)\sim -m^2+3\lambda\int^{\Lambda}\frac{d^3k}{(2\pi)^3}\frac{T}{k^2}=-m^2+\frac{3\lambda}{2\pi^2}\Lambda T. \tag{70}$$

Λ is the ultraviolet cutoff which in many cases has physical meaning because generally scalar field theories are effective low energy models. An example is the scale of separation between the true quantum behavior of excitations at large k^2 and the effective classical dynamics for the long wavelength modes which we describe stochastically. In this case, at high temperature and small coupling λ, $\Lambda\simeq T$.

The interesting fact about $m^2(T)\to 0$ at criticality is that is supplies us with an arbitrarily long time scale separation between the thermalization time for the short and long wavelength modes. As we discussed in Sec. III the long wavelength modes $k^2\simeq 0$ thermalize in a typical time-scale

$$t_{eq}\simeq\frac{\eta}{m^2(T)}\to_{T\to T_c}+\infty, \tag{71}$$

which is the expression of critical slowing down in our stochastic system. On the other hand the short-wave length modes thermalize instead on a characteristic time scale

$$t_{eq}\simeq 2/\eta. \tag{72}$$

It is worth commenting of the form (71). It shows that even in the context of a second order in time Langevin evolution the long wavelength modes are effectively overdamped in the critical domain. This is the essence of the perhaps more familiar TDGL evolution, which is an expansion in the lowest number of relevant field operators that can only be justified rigorously in the critical domain of a second order transition. We see therefore that our Langevin description encapsulates the TDGL dynamics but in addition also applies more generally.

Now, particularly in $D=3$ (as compared to lower dimensions) the temperature corrections to the mass are dominated by the small wave-length modes. Imagine then that the bath temperature is changed to a new value T_f in the vicinity of the phase transition. Over a time $t\sim 2/\eta$ the thermal mass will then adapt to its new small value $m^2(T_f)$. In contrast the long wave-length modes require a much longer time to rethermalize and stay away from thermal equilibrium for a time $t\sim \eta/m^2(T_f)\gg \eta$.

This imbalance is at the heart of our current understanding of the *dynamics* of second order phase transitions and constitutes in particular the essence of the Kibble-Zurek theory of topological defect formation. The familiar response time $\tau(T)$ is here given by Eq. (71). Incidentally the temperature dependence of Eq. (70) also implies that whatever the time behavior of the external bath temperature the dependence of $m^2(T)$ on time will be the same, i.e. for example for a linear external drive of the bath temperature one also obtains $m^2(t)\propto t$, again on time scales $t\gtrsim 2/\eta$.

The consequences of this imbalance are very important. By crossing a second order phase transition at some finite rate τ_Q imposed externally there will be a time in the vicinity T_c when the long-wavelength modes fall out of thermal equilibrium. Because the system always rethermalizes in its small scales first, soft long wavelength non-trivial field configurations can persist for a long time. In particular topological defects can be ''formed'' in this way.

B. Mean-field Fokker-Planck solutions and spinodal instabilities

The onset of spinodal instabilities can be analyzed using the mean-field approximations discussed in Sec. IV.

We begin by analyzing the behavior of these equations when perturbed away from their thermal static solutions. Starting with the overdamped case and writing $A_k(t)^2=A_{0k}^2+\delta a_k$, where A_{0k}^2 is the equilibrium solution, we see that upon linearization in δa_k

$$\partial_t \delta a_k = -2\frac{k^2+m^2+\Delta m^2(t=0)}{\eta}\delta a_k, \tag{73}$$

which has the obvious solution

$$\delta a_k(t)=\delta a_k(t=0)\exp\left[-2\frac{k^2+m^2+\Delta m^2(t=0)}{\eta}t\right]. \tag{74}$$

This solution too shows critical slowing down when of long wave length perturbations if $m^2+\Delta m^2\to 0$.

Exactly the same exercise can be performed for the underdamped case, with the difference that one now needs to diagonalize the matrix in $\delta a_k, \delta b_k, \delta c_k$, resulting from linearizing around equilibrium. The 3 corresponding eigenvalues are $-\eta, -\eta(1\pm\sqrt{1-4(k^2+m^2+\Delta m^2)/\eta^2}$. Each $\delta a_k, \delta b_k, \delta c_k$, is characterized in general by these 3 modes.

Spinodal instabilities may develop when the m^2 becomes negative, $m^2\to -M^2$. The system then responds to this change by increasing $G(t)$. In the overdamped case again the problem is most easy to analyze. The equation for A_k^2 becomes

$$\eta\partial_t A_k^2 = 2A_k^2(k^2-M^2+\Delta m^2(t)-A_k^2 T). \tag{75}$$

If we start in equilibrium $A_{0k}^2=(k^2+m^2+\Delta m^2(t=0))/T$, the term in brackets is initially approximately $-(M^2+m^2)$. Therefore at early times we obtain

$$A_k^2=A_{0k}^2\exp\left[-\frac{M^2+m^2}{\eta}t\right],$$

$$G(t)=G(t=0)\exp\left[\frac{M^2+m^2}{\eta}t\right]. \tag{76}$$

Only $k^2\ll M^2$ modes grow substantially. For large k^2 Eq. (75) becomes rapidly non-linear and mode growth is stopped.

For the underdamped case the analysis is more involved and we give only a sketch below—numerical solutions can be easily produced if more detail is desired. Exponential growth of the type (76) can be generated if the perturbation in C_k^2 is proportional to A_k^2. On the other hand if we insist on starting from thermal equilibrium the instability is triggered by the growth of C_k^2 away from its equilibrium value $C_k^2=0$. From Eqs. (65) we see that at early times

$$C_k^2=-\frac{M^2+m^2}{T}t$$

$$A_k^2=A_{0k}^2-\frac{k^2-M^2}{T}(M^2+m^2)t^2 \tag{77}$$

$$B_k^2=B_{0k}^2+\frac{(M^2+m^2)}{T}t^2.$$

In turn this behavior leads to a growth in $G(t)\sim G(t=0)[1+(M^2+m^2)t^2]$.

For later times when A_k^2 and B_k^2 approach the new thermal fixed point, C_k^2 ceases to grow and decays to zero on a time scale $t\sim 1/\eta$.

The characteristic time scales for the instability to develop are typical for an overdamped response, $t\sim \eta/(M^2+m^2)$, and of the underdamped (relativistic) behavior where $t\sim 1/\sqrt{M^2+m^2}$.

In order to be justified in using a linearized evolution, both directly in the Langevin picture and in the Fokker-Planck context, it is fundamental to place a size restriction on G. Computations of $G(x,t)$ at the point where the linearized approximation fails have been shown to be useful [20], e.g., in the determination of topological defect formation through the insertion of $G(x,t)$ into the well-known Halperin formula for counting field zeros, provided there is little energy left in the system at this point to cause substantial subsequent transport towards short wavelengths (see [1] for counterexamples).

VI. DISCUSSION AND CONCLUSIONS

In this paper I discussed general properties of the second order Langevin scalar field evolution as well as of some of its related models. The associated Fokker-Planck equations was solved in closed form for a harmonic potential and a mean-field approximation was devised for both second order and overdamped cases. The analogous treatment for the stochastic nonlinear Schrödinger equation is given in Appendix A.

Although limited to special cases closed form solutions of the Fokker-Planck equations, have been shown to be very useful in providing us with analytical qualitative (and sometimes quantitative) insights into the evolution of the fully non-linear field theoretical system, especially in determining time scales for thermalization of several parts of the system. Through this analysis I showed how fundamental ingredients of the dynamics of second order phase transitions are embodied in these effective models and how they generalize the well known TDGL evolution in the critical domain.

In particular in the vicinity of T_c, where $m^2(T)\to 0$, one obtains true time scale separation, with the small wavelength field modes thermalizing in a time $t_{eq}\sim \eta^{-1}$, while the long wavelength modes take a time $t_{eq}\sim \eta/m^2(T)\gg 1/\eta$. In these circumstances, over times $1/\eta \ll t \ll \eta/m^2(T)$, one can take the short wavelength modes to be thermalized at the temperature of the stochastic thermal bath, and include their effects on the long-wavelength modes through the temperature dependence of the parameters in their potential. This treatment allows us to discuss in qualitative correct terms the evolution of the long wavelength modes in the critical domain, the phenomenon of critical slowing down and the ingredients of the theory of topological defect formation in these models.

ACKNOWLEDGMENTS

It is a pleasure to thank N. Antunes, S. Habib, G. Lythe and W. H. Zurek for useful comments. This research was supported by the U.S. Department of Energy, under contract W-7405-ENG-36.

APPENDIX A

In this appendix I repeat the analysis applied to the Langevin equations (1) to the case of a SNLS, known in some instances also as the stochastic Gross-Pitaevsii equation. This model is important in the study of non-relativistic systems. Many examples exist that are thought to obey this effective dynamics, ranging from strong type II (or hard) superconductors and superfluid ^{4}He to atomic Bose-Einstein condensates and light propagation in non-linear media.

The sNLS is of the form

$$-(i+\eta)\partial_t\psi = -\nabla^2\psi + \frac{\delta V}{\delta\psi^*} + \xi(x,t), \tag{A1}$$

where $\psi=\psi(x,t)$ is a complex field in space and time. The potential V is usually taken to be

$$V[|\psi|^2] = m^2|\psi(x,t)|^2 + \frac{\lambda}{2}|\psi(x,t)|^4, \tag{A2}$$

where m^2 can be positive or negative. This theory is essentially the non-relativistic version of the second order in time Langevin system for a complex scalar field. The theory is manifestly invariant under global phase $U(1)$ transformations. Its Hamiltonian is

$$H = \int d^Dx\{|\nabla\psi|^2 + V[|\psi|^2]\}. \tag{A3}$$

Although the stochastic terms in Eq. (A1) may be representative of additional intrinsic degrees of freedom, it may also be possible to actually build an experimental situation in which the (many-body) system is driven externally and has losses to the outside so as to realize Eq. (A1). In this latter picture the requirements are that the driving field should be phase incoherent (at least over some short characteristic time and spatial scales) and that the losses would be proportional to the frequency of the excitations in the system. Perhaps the most challenging feature would be to find a trap or mirror whose transmission is linear in the range of frequencies expected in the system.

The Fokker-Planck equation corresponding to this evolution is

$$\partial_t P = \left\{ \frac{1}{i+\eta}\frac{\delta}{\delta\psi}\left[2\frac{\delta H}{\delta\psi^*} + \frac{\Omega}{-i+\eta}\frac{\delta}{\delta\psi^*}\right] + \frac{1}{-i+\eta}\frac{\delta}{\delta\psi^*}\left[2\frac{\delta H}{\delta\psi} + \frac{\Omega}{i+\eta}\frac{\delta}{\delta\psi}\right]\right\} P \tag{A4}$$

where $\Omega = 2\eta T$ as before. The system thermalizes to its canonical distribution

$$P_0 = \mathcal{N}\int D\psi D\psi^* \exp[-\beta H]. \tag{A5}$$

The set of eigenvalues and eigenvectors for the harmonic problem are

$$E_N = 2N\eta\frac{k^2+m^2}{1+\eta^2}, \tag{A6}$$

$$F_N = H_N[\sqrt{\beta(k^2+m^2)\psi_k\psi_k^*}]. \tag{A7}$$

The factor of 2 in E_N relative to Eq. (50) accounts for 2 real fields in the complex quantity ψ, instead of one. Thus, the equilibration time scale is

$$t_{eq} \sim \frac{1+\eta^2}{2\eta(k^2+m^2)}, \tag{A8}$$

which again displays critical slowing down for $k^2\simeq 0$ and $m^2(T)\to 0$.

It is clear that the overdamped limit is recovered for large η, as expected. We therefore see that in the overdamped limit all three models considered in this paper lead to the same characteristic time scale for equilibration as could be expected on general grounds. The differences arise for the short wavelength modes in the system and/or for small η.

An experimental realization of Eq. (A1) would allow for the driving of the physical system across its phase transition and more generally to canonical equilibrium at an arbitrary temperature, by changing the intensity of the phase incoherent driving field ξ. If ξ had a phase coherent component, in addition to the incoherent piece, the drive would make the system behave like an XY magnet at finite temperature in the presence of a magnetic field in the XY plane. This would allow for experiments in non-linear optics or atomic Bose-Einstein condensates to explore regimes similar to those realized in, e.g., high-T_c superconductors.

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